SECTION C — (3 × 10 = 30 marks)

Answer any THREE questions.

16. State and prove the Hahn-Banach theorem.

17. Let $\{e_1, e_2 \ldots e_n\}$ be a finite orthonormal set in a Hilbert space H. If x is any Vector in H, then prove that $\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$. further,

$x - \sum_{i=1}^{n} (x, e_i)\, e_i \perp e_j$ for each j.

18. If T is an operator on H, then prove that the following conditions are all equivalent to one another.

(a) $T^* T = I$

(b) $(Tx, Ty) = (x, y)$ for all x and y.

(c) $\|Tx\| = \|x\|$ for all x.

19. Prove that $r(x) = \lim \|x^n\|^{\frac{1}{n}}$.

20. State and prove Gelfand-Neumark theorem.

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NOVEMBER/DECEMBER 2024

GMA43/DMA43 — FUNCTIONAL ANALYSIS

Time : Three hours Maximum : 75 marks

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SECTION A — (10 × 2 = 20 marks)

Answer ALL questions.

1. What is Banach space?

2. Define bounded linear transformation.

3. State schwarz inequality.

4. Define orthonormal set.

5. When will you say that the self-adjoint operator A is positive?

6. Define projection on M.

7. Write the formula for the spectral radius $r(x)$.

8. Define left regular.

9. What is multiplicative functional?

10. What is Banach* – algebra?

SECTION B — ($5 \times 5 = 25$ marks)

Answer ALL questions.

11. (a) Let N and N^1 be normed linear spaces and Ta linear transformation of N into N^1. Then prove that the following conditions on T are all equivalent to one another :

(i) T is continuous

(ii) T is continuous at the origin in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$.

(iii) There exists a real number $K \geq 0$ with the property that $\|T(x)\| \leq K\|x\|$ for every $x \in N$;

(iv) If $S = \{x : \|x\| \leq 1\}$ is the closed unit sphere in N, then its image T(s) is a bounded set in N^1.

Or

(b) If M is a closed linear subspace of a Normed linear space N and x_0 is a vector not in M, then prove that there exists a functional f_0 in N^* such that $f_0(M) = 0$ and $f_0(x_0 \neq 0)$.

12. (a) State and prove the open Mapping theorem.

Or

(b) If $\{e_i\}$ is an orthonormal set in a Hilbert space H, then prove that $\Sigma|(x, e_i)|^2 \leq \|x\|^2$ for every vector x in H.

13. (a) If T is an operator on H, then prove that T is normal iff its real and imaginary parts commute.

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Or

(b) If P is the projection on a closed linear subspace M of H, then prove that M reduces an operator $T \Leftrightarrow TP = PT$.

14. (a) If G is an open set, and therefore prove that S is a closed set.

Or

(b) If $1 - xr$ is regular, then prove that $1 - rx$ is also regular.

15. (a) $M \to f_M$ is a one-to-one mapping of the set μ of all maximal ideals in on the set of all its multiplicative functionals.

Or

(b) If A is self-adjoint, then $\overline{A}$ is dense in $e(\mu)$.